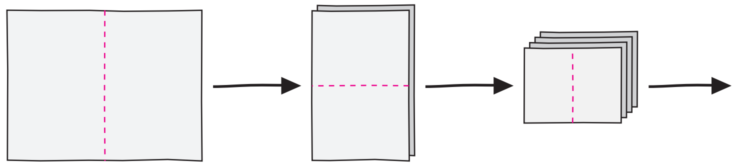
Investigation 1

Exponential Growth

In this investigation, you will explore *exponential growth* as you cut paper in half over and over and read about a very smart peasant from the ancient kingdom of Montarek. You will compare exponential growth with linear growth. You will also explore exponential patterns in tables, graphs, and equations.

1.1 Making Ballots

Chen, the secretary of the Student Government Association, is making ballots for tonight's meeting. He starts by cutting a sheet of paper in half. He then stacks the two pieces and cuts them in half. He stacks the resulting four pieces and cuts them in half. He repeats this process, creating smaller and smaller pieces of paper.



After each cut, Chen counts the ballots and records the results in a table.

Number of Cuts	Number of Ballots
1	2
2	4
3	
4	

Chen wants to predict the number of ballots after any number of cuts.

Problem 1.1 Introducing Exponential Relationships

A. Make a table to show the number of ballots after each of the first five cuts.

B. Look for a pattern in the way the number of ballots changes with each cut. Use your observations to extend your table to show the number of ballots for up to 10 cuts.

C. Suppose Chen could make 20 cuts. How many ballots would he have? How many ballots would he have if he could make 30 cuts?

D. How many cuts would it take to make enough ballots for all 500 students at Chen's school?

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1.2 Requesting a Reward

When you found the number of ballots after 10, 20, and 30 cuts, you may have multiplied long strings of 2s. Instead of writing long product strings of the same factor, you can use **exponential form.** For example, you can write $2 \times 2 \times 2 \times 2 \times 2$ as 2^5, which is read "2 to the fifth power."

In the expression 2^5, 5 is the **exponent** and 2 is the **base.** When you evaluate 2^5, you get $2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32$. We say that 32 is the **standard form** for 2^5.

Getting Ready for Problem 1.2

- Write each expression in exponential form.
  - **a.** $2 \times 2 \times 2$ **b.** $5 \times 5 \times 5 \times 5$
  - **c.** $1.5 \cdot 1.5 \cdot 1.5 \cdot 1.5 \cdot 1.5 \cdot 1.5 \cdot 1.5$
- Write each expression in standard form.
  - **a.** 2^7 **b.** 3^3 **c.** 4.2^3
- Most calculators have a [^] or [y^x] key for evaluating exponents. Use your calculator to find the standard form for each expression.
  - **a.** 2^{15} **b.** 3^{10} **c.** 1.5^{20}
- Explain how the meanings of 5^2, 2^5, and 5×2 differ.

One day in the ancient kingdom of Montarek, a peasant saved the life of the king's daughter. The king was so grateful he told the peasant she could have any reward she desired. The peasant—who was also the kingdom's chess champion—made an unusual request:

"I would like you to place 1 ruba on the first square of my chessboard, 2 rubas on the second square, 4 on the third square, 8 on the fourth square, and so on, until you have covered all 64 squares. Each square should have twice as many rubas as the previous square."

The king replied, "Rubas are the least valuable coin in the kingdom. Surely you can think of a better reward." But the peasant insisted, so the king agreed to her request. *Did the peasant make a wise choice?*

Problem 1.2 Representing Exponential Relationships

A. **1.** Make a table showing the number of rubas the king will place on squares 1 through 10 of the chessboard.

2. How does the number of rubas change from one square to the next?

B. Graph the (*number of the square, number of rubas*) data for squares 1 to 10.

C. Write an equation for the relationship between the number of the square n and the number of rubas r.

D. How does the pattern of change you observed in the table show up in the graph? How does it show up in the equation?

E. Which square will have 2^{30} rubas? Explain.

F. What is the first square on which the king will place at least one million rubas? How many rubas will be on this square?

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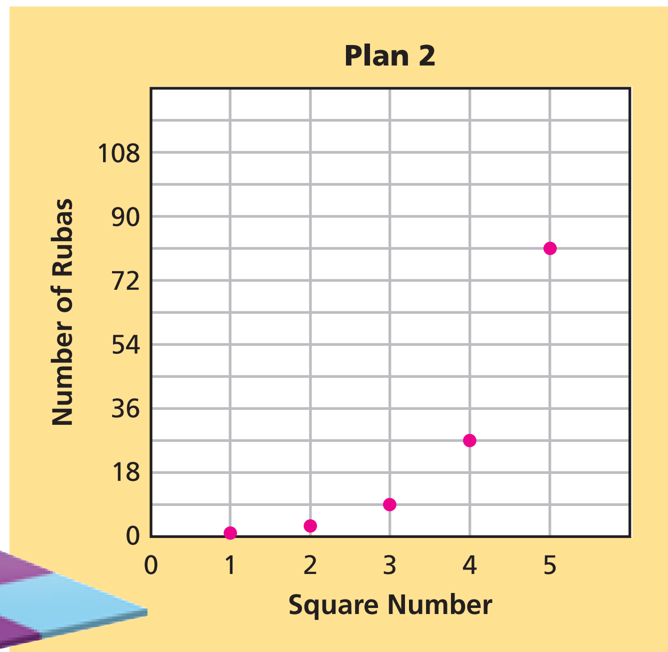
1.3 Making a New Offer

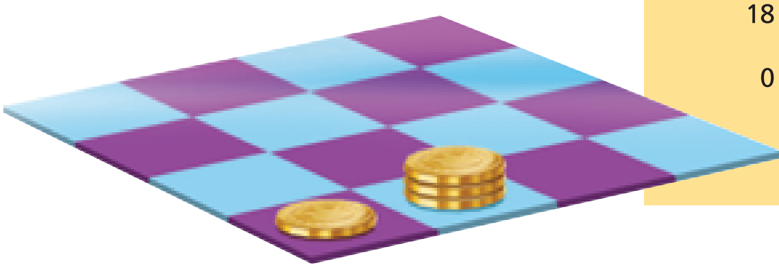
The patterns of change in the number of ballots in Problem 1.1 and in the number of rubas in Problem 1.2 show **exponential growth.** These relationships are called **exponential relationships.** In each case, you can find the value for any square or cut by multiplying the value for the previous square or cut by a fixed number. This fixed number is called the **growth factor.**

- What are the growth factors for the situations in Problems 1.1 and 1.2?

The king told the queen about the reward he had promised the peasant. The queen said, "You have promised her more money than we have in the entire royal treasury! You must convince her to accept a different reward."

After much thought, the king came up with Plan 2. He would make a new board with only 16 squares. He would place 1 ruba on the first square and 3 rubas on the second. He drew a graph to show the number of rubas on the first five squares. He would continue this pattern until all 16 squares were filled.





The queen wasn't convinced about the king's new plan, so she devised a third plan. Under Plan 3, the king would make a board with 12 squares. He would place 1 ruba on the first square. He would use the equation $r = 4^{n-1}$ to figure out how many rubas to put on each of the other squares. In the equation, r is the number of rubas on square n.

Problem 1.3 Growth Factors

A. In the table below, Plan 1 is the reward requested by the peasant. Plan 2 is the king's new plan. Plan 3 is the queen's plan. Copy and extend the table to show the number of rubas on squares 1 to 10 for each plan.

Reward Plans

Square Number	Number of Rubas		
	Plan 1	Plan 2	Plan 3
1	1	1	1
2	2	3	4
3	4	■	■
4	■	■	■

B. 1. How are the patterns of change in the number of rubas under Plans 2 and 3 similar to and different from the pattern of change for Plan 1?

2. Are the growth patterns for Plans 2 and 3 exponential relationships? If so, what is the growth factor for each?

C. Write an equation for the relationship between the number of the square n and the number of rubas r for Plan 2.

D. Make a graph of Plan 3 for $n = 1$ to 10. How does your graph compare to the graphs for Plans 1 and 2?

E. The queen's assistant wrote the equation $r = \frac{1}{4}(4^n)$ for Plan 3. This equation is different from the one the queen wrote. Did the assistant make a mistake? Explain.

F. For each plan, how many rubas would be on the final square?

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1.4 Getting Costs in Line

Before presenting Plans 2 and 3 to the peasant, the king consulted with his financial advisors. They told him that either plan would devastate the royal treasury.

The advisors proposed a fourth plan. Under Plan 4, the king would put 20 rubas on the first square of a chessboard, 25 on the second, 30 on the third, and so on. He would increase the number of rubas by 5 for each square, until all 64 squares were covered.

To help persuade the peasant to accept their plan, the advisors prepared the following table for the first six squares. The king presented the plan to the peasant and gave her a day to consider the offer.

Reward Plans

Square Number	Number of Rubas	
	Plan 1	Plan 4
1	1	20
2	2	25
3	4	30
4	8	35
5	16	40
6	32	45

Do you think the peasant should accept the new plan? Explain.

Problem 1.4 Comparing Growth Patterns

A. Is the growth pattern in Plan 4 an exponential relationship? Explain.

B. Describe the graph of Plan 4 and compare it to the graph of Plan 1.

C. **1.** Write an equation for the relationship between the number of the square n and the number of rubas r for Plan 4.

2. Compare this equation to the equation for Plan 1.

3. How is the change in the number of rubas from one square to the next shown in the equations for Plan 1 and Plan 4?

D. For Plans 1 and 4, how many rubas would be on square 20? How many rubas would be on square 21?

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